A molecular dynamics study of non-local effects in the flow of soft jammed particles

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In this paper, we used numerical simulations to investigate the flow properties of soft glassy materials. These systems display a mixed fluid–solid behavior whose theoretical description remains a challenging task. The molecular dynamic simulations exhibit non-local rheological behavior, in direct line with previous experimental results. The inverse viscosity of the material at a given point, denoted as fluidity, is not a local function of the local stress, but also depends on the state of the system in the neighborhood, with a spatial correlation length typically equal to a few particles. The fluidity is furthermore related directly to the velocity fluctuations and rate of plastic events in the form of a scaling function. Correlations are the signature of a cooperative process at the origin of the flow and of the non-local effects. We compare the obtained results with a scalar fluidity model and emphasize the similarities between the two approaches.

1 Introduction

Soft glassy materials encompass polymers, foams, concentrated emulsions, concentrated solutions of colloids and granular assemblies. Because of the existence of short-range forces, steric forces, and polydispersity, the microscopic components of the soft glassy materials are caged by their neighbors and their structures remain frustrated and disordered. These systems display complex flow properties, intermediate between that of a solid and a liquid: at rest they behave like an elastic solid, but flow “like a liquid” under sufficient applied stress. This mixed fluid–solid behavior occurs above a threshold volume fraction associated with the appearance of a yield stress . The yielding behavior makes such systems particularly interesting for applications from toothpaste to coatings, cosmetic and food emulsions but remains fundamentally difficult to describe.

At this stage, a global picture describing the mechanisms at the origin of the flow starts to emerge. In these jammed systems, flow occurs through a succession of elastic deformations and local irreversible plastic rearrangements associated with a microscopic yield stress. These localized events induce long-range elastic modifications of the stress over the system, thereby creating long-lived fragile zones where flow occurs. Flow in these systems is thus highly cooperative and spatially heterogeneous: a dynamically active region will induce stress fluctuations of its neighbors and thus a locally higher rate of plastic rearrangements. Correlations between plastic events are accordingly expected to exhibit a complex spatio-temporal pattern.

This picture is supported by recent experiments on various systems, emulsions, foams and granular materials. Non-locality is furthermore evidenced by several intriguing behaviors of jammed materials. As reported by experiments carried out by Van Hecke’s group, a steel ball resting on a bucket of sand was observed to sink as soon as a shear band was created at the bottom of the bucket, far away from the wall. This observation suggests that the presence of a sheared region somewhere in the granular medium modifies the rheological properties of the sample everywhere. In another configuration using Couette geometry, Reddy et al. studied the influence of a shear band close to the inner cylinder on the motion of an intruder far from the band. A rod plunged into the granular material at a distance from the inner cylinder is moved by imposing an orthogonal force . In the absence of shear bands, a minimum force has to be applied in order to move the intruder. As soon as the shear-band is created, the rod moves for any applied force. Stress release and fluctuations induced by localized flow help the granular medium to yield. Finally, experiments on concentrated emulsions in a confined Poiseuille flow using microfluidic devices showed a strong departure from the expected bulk behavior. In this study, a non-zero shear rate, with non-vanishing fluctuations of the local shear rate and plastic rearrangements, is observed at the center of the channel, a region where the shear stress is vanishing and the material is not expected to flow. Furthermore the local strain rate fluctuation, which reflects the number of plastic events by unit of time, is found to be proportional to the local fluidity, defined as the inverse of the viscosity (i.e. the local shear rate

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divided by the local shear stress). It is spatially correlated and exhibits non-locality. The rate is higher in the vicinity of an active zone, as compared to its value near a quieter one. It was argued that this cooperative behavior is due to the non-local elastic relaxation of the local shear stress after a plastic event, which induces an increase of the local stress in a zone close to an event. Velocity profiles in confined geometry were shown to strongly depart from the bulk rheology predictions. They are influenced by the nature of the boundary conditions and by the confinement.

Such features appear therefore to be generic for a large class of materials, suggesting an underlying common flow scenario. At this stage, the theoretical description of those behaviors remains a challenging task. Fully microscopic works using statistical physics are, up to now, limited to model monodisperse systems. Progress has been made however on the basis of mesoscopic approaches. Different models address the rheology question among which one may quote: Soft Glassy Rheology (SGR), Shear Transformation Zones (STZs), the Non-Local (NL) model by Pouliquen and Forterre, fluidity models (FMs) and the Kinetic Elasto Plastic (KEP) model. All these models describe plastic events as the yielding of some elements. They introduce the distribution of stress (SGR, NL, KEP), local fragility (STZ) or of local fluidity (FM) as a key physical quantity. Mechanical noise is described in terms of an effective temperature (SGR, STZ) or via a self-consistent activated process (NL, FM, KEP). The NL and KEP approaches lead to constitutive laws that can be easily and quantitatively compared with experimental data. They both evidence non-local rheology, i.e. dependence of the viscosity at a given point on the local stress but also on the dynamical state in the neighborhood of the point.

In order to validate the pertinence of rheological models, it is crucial to obtain information on the dynamical processes occurring in these materials at all scales. Numerical simulations are accordingly a powerful tool to understand the evolution of soft glassy materials; they explicitly allow the micro–macro link to be made, while maintaining the full complexity of the materials. While some progress has been made recently to develop mesoscopic large-scale simulations of yielding materials, here we will use molecular dynamics simulation of a model emulsion introduced by Durian. In recent times, this model has been extensively studied via numerical simulations to study various dynamical properties of soft jammed particles (like emulsions and foams).

Recently, this model has been also used to successfully reproduce the confinement-dependent dynamical arrest of gravity driven flows of dense foams, as observed by Lespiat et al.

In this configuration, Chaudhuri et al. showed that the threshold forcing strongly depends on the width of the channel, with the threshold values for narrower channels being increasingly larger than the bulk yielding value. This result echoes experiments in granular materials, where the flow arrest of thin layers down inclined planes requires a larger tilt angle (i.e. larger stress) for thinner layers. In this article, we aim at comparing simulation results to the prediction of actual theoretical descriptions. We focus in particular on the non-local KEP fluidity model for the theoretical descriptions. We will compare both approaches in three geometries: periodic Couette flow, confined Couette flow and confined Poiseuille flow. The first part is devoted to the description of the numerical simulation implementation. The second part of this article recalls the main results of the non-local KEP fluidity model. In the third part, we display the rheological properties in periodic geometries. The fourth part discusses flow in a confined Couette flow, while the fifth part explores the flow in a confined Poiseuille flow. The last part is devoted to an analysis at particle scale focused on the rearrangement rate and on the velocity fluctuations.

2 Numerical simulations

2.1 Interparticle forces

To simulate the flow of soft glassy materials, we use molecular dynamic simulations with the simple model of Durian for interparticle interaction. The system is a collection of athermal polydisperse soft disks interacting via elastic repulsion and viscous dissipation. This model is simple enough to enable relatively long simulations but contains most of the elements of a jammed system. Here, we study a two-dimensional system of such particles, in order to reduce the calculation time and improve statistics. The disks have a random radius uniformly distributed between $R_c(1 - \sqrt{3}\delta)$ and $R_c(1 + \sqrt{3}\delta)$, where $R_c$ is the mean radius and $\delta$ is the polydispersity. The volume fraction $\phi$ is tuned by varying the number of particles.

Two particles $i$ and $j$ interact via an elastic repulsion and a viscous dissipation only if they are in contact, i.e. if $\Delta r_{ij} = (R_i + R_j) - \left| \vec{r}_i - \vec{r}_j \right|$ where $R_i$ is the radius of the particle $i$. The elastic and viscous forces are chosen as:

$$\vec{F}_{e,i,j} = -k\Delta r_{ij}\hat{n}_{ij}$$

$$\vec{F}_{v,i,j} = -b((\vec{v}_i - \vec{v}_j)n_{ij}\hat{n}_{ij}$$

with

$$\hat{n}_{ij} = \frac{\vec{r}_j - \vec{r}_i}{\left| \vec{r}_j - \vec{r}_i \right|}$$

The particles have a constant mass $m_i$ and obey the Newton’s equation of motion:

$$m_i \frac{d^2\vec{r}_i}{dt^2} = \sum_j \vec{F}_{e,i,j} + \vec{F}_{v,i,j} + \vec{F}_{ext}$$

where $\vec{F}_{ext}$ is an external forcing added only for a Poiseuille flow and will be discussed later. The flow is implemented by integrating this set of equations using the Verlet algorithm.

In the original study of Durian (where the inertia of the particles was neglected), it has been shown that the rheological properties of such a system of particles, simulated in wide geometry, are well described by a Herschel–Bulkley law $\sigma = \sigma_y + A\gamma^n$ with a coefficient $n = 1$ in 2D and in 3D. In a later work, using a model similar to ours, Langlois et al. found a coefficient $n = 1/2$, which has also been observed in recent simulations.
2.2 Unit and parameters

This model requires 4 parameters — $R_s$, $\kappa$, $b$ and $m_s$ — and two dimensionless parameters: the polydispersity $\delta$ and the volume fraction $\Phi$. Using dimension consideration, we can set three units of the model: the length unit $R_s$, the time unit $\tau_s = \sqrt{m_s/\kappa}$ and the energy unit $\epsilon = m_s R_s^2/\tau_s^2$. For simplicity the friction coefficient $b$ is taken equal to $b = 1 \times \epsilon \tau_s/R_s^2$.

We have explored several surface fractions between 90% and 110% (above the jamming fraction, 84.5%, for a 2D system) and polydispersity from 0.2 to 0.4. If the flow curve depends on the surface fractions [Fig. 2], the observations on the non-local effects are not affected by these two parameters. For this reason and for the sake of clarity most of our results are presented for a surface fraction $\Phi$ equal to 100% and a polydispersity of $\delta = 0.35$. These parameters are the same used by Chaudhuri et al.\textsuperscript{37}

The system size is typically $10^3$ to $10^4$ particles.

2.3 Bulk shear Couette flow

An ‘infinite’ Couette flow is simulated using periodic boundary conditions. We use a box size of typically $120R_s$. We first prepare the sample in a static periodic boundary condition. We spread the particles randomly and uniformly, with a diameter ten times each particle is applied in opposite directions ($\pm F_e^s$). The system is split in the $x$-direction, by imposing an external stress, is created by applying a constant force, $F^{\text{ext}} = F^{\text{ext}} \epsilon x$, along the $x$-direction on the top layer, while the lower wall is kept static.\textsuperscript{29,37} The volume is kept constant by fixing the top wall in vertical position $y$. Periodic boundary conditions are imposed in the flow direction, $x$. After a typical time $10^3 \tau_s$, the system reaches a steady state and the shear stress $\sigma(x,y)$ does not depend upon the spatial position. It is homogeneous and equal to $\sigma(x,y) = \Sigma = F^{\text{ext}}/L_x$. We have also realized simulations at an imposed shear rate. In that case the velocity of the top wall is set constant and the bottom wall remains static. The stress measured with the same methods as for the bulk flow simulations is found to be constant.

To create a confined Poiseuille flow, we prepare a system similar to a confined Couette flow. In this case, each of the particles, confined by the static walls made of the frozen particles, is subjected to a constant external force $F^{\text{ext}} = F^{\text{ext}} \epsilon x$; this generates a Poiseuille flow in the $x$-direction.

The local shear stress $\sigma(x,y)$ depends on the $y$ position, it is imposed by momentum conservation: $\sigma(x,y) = \rho F^{\text{ext}} |y|/\pi (R^2)$. Shear stress can also be calculated from the virial model. Note that both calculations lead to the same value in the following simulations. To test the impact of the walls’ roughness on the flow, we have also performed a simulation with smooth confining walls. In that case the walls are not made of the frozen particles, but each of them is perfectly smooth and interacts with the particles via the same elastic repulsion and viscous dissipation. We also show some results for the case of periodic Poiseuille flow, which has been recently studied by Chaudhuri et al.\textsuperscript{37} In this case, a channel of width $2w$ is split in the $y$-direction into two subdomains each of width $w$. In each such subdomain, the body force $F^{\text{ext}}$ on each particle is applied in opposite directions ($\pm \epsilon x$), with periodic boundary conditions being applied in both $x$ and $y$ directions. Such a construction allows for spatially homogeneous densities even for small channel-widths. For all these geometries we have measured the local shear stress using the same methods as in the previous section. A very good agreement is found between the measurement and the predictions.

3 Fluidity models

3.1 Fluidity models and kinetic descriptions

In this section we recall the main ingredients of fluidity models, to which we will compare the results of our numerical simulations. A detailed version of the model is given in the Appendix. The essence of this class of models is to describe the local state of the system by a single scalar variable, the rate of plastic events $\Gamma$. We anticipate that the rate of plastic events is inversely proportional to the local viscosity. The evolution of the rate of plastic events results from the competition between the relaxation of dissipative systems towards a quiescent, disordered state and flow rejuvenation.\textsuperscript{22,23} The rate of plastic events in the system decreases during the spontaneous evolution of the pasty system, the system becoming slower and slower with time. By contrast, flow induces rearrangements which act as a seed for new events and thus increase $\Gamma$ as a feedback mechanism.

Following this picture, a kinetic model for the elastoplastic dynamics of a jammed material has been proposed (hereafter coined as the ‘KEP model’), which takes the form of a nonlocal – Boltzmann-like – kinetic equation for the stress distribution function, in line with the pioneering work of Hébraud and Lequeux.\textsuperscript{31} In the steady state, under the hypothesis of low cooperativity the model predicts rheological equations which take the generic form described above:\textsuperscript{24}

$$\sigma = G(6\Gamma)^{-1} = \gamma f_f$$
\[
\Delta G - \frac{1}{3}(G - G_b) = 0
\]  

where \( G_0 \) is the bulk elastic modulus, \( \xi \) the cooperativity length assumed independent of the shear stress in this crude approach, and \( f = 6\xi / G_0 \) the fluidity. \( T_b \) is the value of the plastic events rate in the absence of spatial heterogeneities i.e. in wide geometry under a homogeneous shear stress. It writes:

\[
\sigma \leq \sigma_d : T_b = 0
\]

\[
\sigma > \sigma_d : T_b = \left( \frac{1}{\xi \sigma} \right) \times (\sigma - \sigma_d)^2
\]

where \( \sigma_d \) is the dynamic yield stress and \( \xi \) is the characteristic time. As a side remark, we note that in wide homogeneous geometry, the fluid obeys a Herschel–Bulkley law \( \sigma = \sigma_d(1 + (\xi \gamma)^{1/2}) \).

### 3.2 Boundary conditions

The resolution of fluidity equations, e.g. eqn (17), requires boundary conditions. These conditions are fixed by the specific behavior at the walls. How the wall affects the density, order and relaxation processes of soft glassy materials in their vicinity is obviously a complex question even for simple liquids.\textsuperscript{31} A smooth surface can induce ordering, while roughness may locally disorganize the fluid (relative to the bulk behavior). The consequences of these factors on the local fluidity are far from trivial and we leave these questions for future work. Here we choose to study the simplified case of an imposed fluidity (or a fixed rate of plastic events) at the wall, in the form:

\[
f_{\text{surf}} = f_{\text{wall}}(\sigma_{\text{wall}})
\]

or the equivalent relation for the rate of plastic events, \( f_{\text{surf}} = f_{\text{wall}}. \) In the numerical simulations, we will extract directly the wall fluidity \( f_{\text{wall}}(\sigma_{\text{wall}}) \) from the numerical computation by writing that \( f_{\text{wall}} = \sigma_{\text{wall}} / \gamma_{\text{wall}} \). A similar approach was used previously to compare the KEP model to experiments in emulsions.\textsuperscript{9}

In doing so, we can disentangle the questions of non-local rheology, which is the main focus of the present work, from the detailed understanding of boundary conditions for a flowing jammed material at a solid surface. We leave the latter question for future work, specifically dedicated to the microscopic mechanisms of surface flows.

### 4 Rheological behavior in homogeneous Couette geometry

We start by considering flow in the periodic planar Couette flow, following Lees and Edwards’ procedure, as detailed in Section 3. The resulting velocity profile is linear with a well defined shear rate \( \dot{\gamma} \). The resulting flow curve, relating the mean shear stress to the imposed shear rate, is presented in Fig. 1.

The stress is a monotonic function of the shear rate \( \dot{\gamma} \) and evidences a yield stress. Furthermore, in the parameter range studied, the shear stress obeys a Herschel–Bulkley law

\[
\sigma = \sigma_d(1 + (\xi \gamma)^{1/2}) \text{ with } \sigma_d = 0.0058 \kappa \text{ and } \xi = 350 \tau_c.
\]

\[
\sigma = \sigma_d(1 + (\xi \gamma)^{1/2}) \text{ with } \sigma_d = 0.0058 \kappa \text{ and } \xi = 350 \tau_c.
\]

\[
\sigma = \sigma_d(1 + (\xi \gamma)^{1/2}) \text{ with } \sigma_d = 0.0058 \kappa \text{ and } \xi = 350 \tau_c.
\]
surface fraction modifies it. As expected the dynamical yield stress increases as a function of the surface fraction whereas the characteristic time decreases.

5 Rheological properties in confined planar Couette geometry

5.1 Numerical simulations

To probe non-local effects, we now explore confined geometry and first focus on the confined planar Couette cell geometry, i.e. shearing of the material between two identical parallel walls separated by a distance \( w \). Details of the simulation are given in part 2.

Below the yield stress no flow is observed. When the stress is higher than the yield stress the system starts to flow and reaches a well established steady state. The velocity profiles present then the same features whatever the stress. No shear banding is observed.

Fig. 3 displays the reduced velocity profiles at a given stress \( \Sigma = 2 \sigma_d \), for various confinements. The velocity has been divided by \( w \dot{\gamma}_{\text{bulk}} \) where \( \dot{\gamma}_{\text{bulk}} \) is the value of the shear rate associated with \( \Sigma \) obtained from the ‘bulk’ flow curve measured using the Lees–Edwards’ procedure. In the absence of non-local effects, one would expect the reduced velocity profiles to be a straight line of slope unity in this representation. This is not the case. Near the wall, the shear rate is higher than the bulk value. It decreases towards the bulk value over a distance of about 5\( R_s \). For a strong confinement (\( w = 15 R_s \)), the shear rate cannot reach its bulk value at the center. The non-linearity of the velocity suggests that the local flow curve, relating \( \dot{\gamma}(y) \) to \( \sigma(y) \), is non-unique, and does not match perfectly with the bulk flow curve obtained in the wide (periodic) cell from the Lees–Edwards’ procedure. This is the first hint towards the existence of non-local effects.

To quantify those effects, we extract a local flow curve from the velocity profiles. We recall that the slope of the velocity profiles is the local shear rate and that in such a geometry the shear stress is homogeneous and equal to \( \Sigma = f / l_s \). The data are reported in Fig. 4. In line with Fig. 3, several shear-rates correspond to a single stress and data point to a lack of a global flow curve in confined geometry.

We have also realized simulation in a confined Couette flow at an imposed shear rate. In these simulations we impose the velocity of the top plate i.e. we only impose the mean shear rate. So at a steady state the stress is constant inside the gap but not the shear rate. We observed the same velocity profiles as for imposed shear stress: the shear rate is higher close to the wall than at the center.

Finally we inspect the impact of the flow on the local surface fraction. Fig. 5 shows the typical local concentration profile. The concentration is only modified close to the wall where we observe two maxima corresponding to a stratification of the two first layers of particles. The concentration does not depend on the stress. Far from the wall the concentration reaches a

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**Fig. 3** Reduced velocity profiles \( (V - V_{\text{mid}}) / (w \dot{\gamma}_{\text{bulk}}) \) as a function of the reduced position \( y / w \) in confined Couette geometry. (Symbols) Results from simulations for different confinements: from top to bottom \( w = 40 R_s, 30 R_s, 20 R_s, 15 R_s \). (Full lines) Fits of the velocity profiles using \( \xi = 5 R_s \) and \( f_{\text{wall}} = 1.6 f_{\text{bulk}} \) (Dashed lines) Straight lines with a slope of 1. They emphasize that the shear rate is greater at the wall than at the center of the channel. For the sake of clarity, the curves have been arbitrarily shifted vertically.

**Fig. 4** Flow curve obtained in confined Couette geometry. (Symbols) Flow curve extracted from the profiles for three different confinements: (red ○) \( w = 15 R_s \), (blue □) \( w = 30 R_s \), and (green ⋆) \( w = 40 R_s \). (Line) Fit using a Herschel–Bulkley model for the Lees–Edwards’ simulation [Fig. 1].

**Fig. 5** Mean local surface fraction profile in a confined Couette flow for a gap of 30\( R_s \), and two stresses 2\( \sigma_d \) (○) and 3\( \sigma_d \) (□).
constant value equal to the equilibrium value (100%) and the order due to the wall disappears. The flow does not induce migration of particles.

5.2 Comparison to the KEP fluidity model

We now turn to the fluidity model in this confined geometry. In this case the mean shear stress is spatially homogeneous, \( \sigma(\vec{x}) = \Sigma \). We will use the simplified model assuming low cooperativity. Under this hypothesis, the system is governed by eqn (17), for which the resolution is straightforward. Let us define \( x \) as the flow direction and \( y \) as the transverse direction, \( y = 0 \) is defined as the channel center. The rearrangement rate profile is symmetric with respect to the center, and we get

\[
I = A \cosh(\frac{y}{\xi}) + I_b. \tag{17}
\]

The coefficient \( A \) is fixed by the boundary conditions, eqn (7), \( I(\pm w/2) = I(\pm \Sigma) \), and we get:

\[
I(y) = (I_w - I_b) \frac{\cosh(y/\xi)}{\cosh(w/2\xi)} + I_b \tag{18}
\]

with the fluidity directly connected to \( I(y) \) as \( f(y) = 6G_{\infty}^{-1} \times I(y) \). Typical profiles for the rearrangement rate and velocity profiles are shown in Fig. 6.

Several remarks can be made on the basis of these results. First, the predictions for the velocity profiles exhibit features in qualitative agreement with the simulation results in Fig. 3: velocity profiles are not straight lines even though the shear stress is homogeneous, and their shape exhibits a dependence on confinement. Deviations from the linear profiles close to the wall extend over a distance fixed by \( \xi \).

When the gap is large enough, the reduced velocity reaches the straight line of slope 1 at the center and the fluidity at the center is equal to the bulk fluidity. For a strong confinement, the rearrangement rate does not reach its bulk value at the center. We quote that similar results do not depend on the detailed choice for the boundary conditions.

To go further, a more quantitative comparison of the numerical simulation results in Fig. 3 can be made with the fluidity model. To do so, the bulk fluidity is deduced from the ‘bulk’ data obtained using the Lees–Edwards’ procedure. Then the wall fluidity is extracted from the numerical profile from the measured value of the wall shear-rate for each condition, \( \Gamma_{\text{wall}} = \dot{\gamma}_{\text{wall}}/\Sigma \). In the simulations, the wall fluidity can be well fitted by a simple relationship \( \Gamma_{\text{wall}} = 1.6\Gamma_{\text{bulk}} \) whatever the confinement (not shown). We quote that this behavior is in agreement with the experimental results of Goyon et al. for confined emulsion flows.\(^9\) In the following, we will use this functional dependence as a boundary condition.

Then the fluidity and shear rate are deduced by using eqn (8) with these values. As shown in Fig. 4, a good agreement is then found between simulations and the fluidity predictions, using a single constant value for the cooperative length \( \xi = 5R_0 \). This is the first successful test for the fluidity description.

6 Rheological properties in Poiseuille geometry

6.1 Numerical simulations

We now move to the Poiseuille flow geometry. In contrast to the Couette cell, the stress is not homogeneous over the system. Simulation details are given in Section 2. Fig. 7 displays the corresponding velocity profiles obtained for various external forcing. As the top and the bottom layer of droplets are frozen, the velocity vanishes at the wall and no slip is evidenced.

It is first interesting to compare these velocity profiles to the profiles predicted on the basis of the bulk rheological law, as measured using Lees–Edwards’ simulation. These bulk predictions are displayed in Fig. 7(a): a strong discrepancy is found between the simulation results and the bulk predictions for any forcing. Interestingly, the fluid behaves as if it would be less viscous in the Poiseuille flow than in an infinite Couette cell.

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**Fig. 6** (a): Dimensionless rearrangement rate \( I/I_{\text{bulk}} \) profiles obtained in confined geometry using the KEP model for \( \sigma = 3.5\sigma_{\infty} = 0.482 \). From top to bottom the profiles correspond to \( w = 8\xi, 10\xi, 20\xi, 30\xi, 40\xi \). (b) Dimensionless velocity profiles \( (V - V_{\text{wall}})/(w\dot{\gamma}_{w_{\text{bulk}}}) \) as a function of the reduced position \( y/w \) corresponding to the fluidity profiles displayed in (a) calculated with the KEP fluidity equation (blue full line). The black dotted lines have a slope of 1. For the sake of clarity, the curves have been arbitrarily shifted vertically. We used \( \Sigma = 5\sigma_4 \) and the wall fluidity was assumed to be \( \Gamma_{\text{wall}} = 1.6\Gamma_{\text{bulk}} \). From top to bottom the profiles correspond to \( w = 40\xi, 30\xi, 20\xi, 10\xi, 8\xi \).

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**Fig. 7** (a) (Symbols) Velocity profiles measured from MD simulations in a Poiseuille flow for a channel width \( w = 60R_0 \), and under various forcings. As velocity profiles are symmetric, only half profiles are shown for clarity. From top to bottom the forcing increases with the stress at the wall with \( \sigma_w = 3.25\sigma_4, 3\sigma_4, 2.75\sigma_4, 2.5\sigma_4, 2.25\sigma_4, 2\sigma_4 \). (Lines) Velocity profiles using the Herschel–Bulkley law measured from the Lees–Edwards’ method for the same stresses. These velocity profiles underestimate the one obtained using numerical simulations. Discrepancies larger than a factor of 1.5 are evidenced at a high shear stress. (b) Same as the previous figure, but lines are predictions using the KEP fluidity model. The velocity field can be predicted using the non-local model with the same correlation length as for the Couette flow \( \Sigma = 5R_0 \), a wall fluidity between 1.5 and 2.5\( \Gamma_{\text{bulk}} \), which is consistent with the value measured in Couette geometry and with the bulk fluidity obtained from the Lees–Edwards’ method.
Following the same procedure as for the Couette cell, we also construct the local flow curve as displayed in Fig. 8. We recall that for a constant forcing, the stress is a linear function of the distance to the center; the shear rate $\dot{\gamma}$ is measured from the slope of the velocity profile. Clearly, data are scattered in the whole figure and do not collapse on a single rheological curve and differ from the bulk flow curve measured in the periodic cell. Different flow curves are obtained for different forcing and for the same shear stress variations of more than a factor 1.5 of the shear rate are measured at a high shear rate. This corroborates the fact that the profiles predicted by the bulk curve do not capture the behavior in confined Poiseuille geometry. Also, we observe flow below the yield stress $\sigma_d$, which suggests that the flow near the walls helps the material to yield at the center. For completeness, we also show similar flow curves measured in a three-dimensional Poiseuille flow (see Fig. 9); here too, deviations from the bulk flow curve for the local flow curves (for different imposed forcings) can be seen. Such behavior echoes the experimental observations using concentrated emulsions.

Altogether the simulation results demonstrate the existence of strong finite size effects and non-locality in the Poiseuille flow of the system of jammed particles.

Another indication of non-locality is the influence of wall corrugation on the shape of the flow profile. We have compared the previous results to velocity profiles obtained for a system confined by smooth walls. As expected, for smooth walls, we observe slipping at the surface. However, removing the slip contribution of the velocity by plotting $V - V_{\text{max}}$ where $V_{\text{max}}$ is the maximum velocity at the center, we still find a discrepancy between the two profiles for the rough and smooth walls, see Fig. 10. In the absence of non-locality, the two curves should overlap whatever the slip, which we do not observe. The same material flowing near smooth surfaces appears more viscous than when it flows near rough walls. This effect is also in agreement with the experimental observation by Goyon et al. It is a signature of non-locality.

Finally to conclude this section, we have checked that the non-local behavior is not due to a coupling between the concentration and the flow. Fig. 11 shows the typical local surface fraction in a Poiseuille flow for two surfaces - smooth and rough. As observed for the confined Couette flow simulation, the flow does not induce migration of particles. The concentration profile is only modified close to the wall. For both surfaces we observe two maxima showing a stratification of the two first layers. The maxima of the smooth surfaces are more important, indicating that the stratification is stronger for a smooth surface. Further from the surface the concentration reaches the equilibrium value of 100%.

### 6.2 Comparison to the KEP fluidity model

We now turn to the prediction of the fluidity model in this geometry. In the Poiseuille flow, the flow is created by a drop of pressure $\Delta P$ between the inlet and outlet of a channel with a length $L$. The stress varies spatially in the confined channel according to $\sigma(y) = \frac{\Delta P}{L} y$. The coordinate $y$ displays the distance from the center of the channel and varies between $-w/2$ and $w/2$, where $w$ is the width of the channel. The spatial variation of

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**Fig. 8** Flow curves of the particles extracted from the velocity profiles displayed in Fig. 7. Same symbols as in Fig. 7. The line corresponds to the bulk flow curve obtained in wide geometry using Lees–Edwards’ procedure.

**Fig. 9** Flow curves obtained in three-dimensional simulations of Poiseuille flow, in a channel of width $w = 20a$, of the soft particles having a volume fraction of $\phi = 0.85$ and a polydispersity of $\delta = 0.23$. The different curves correspond to different imposed stresses: (from top to bottom) $\sigma_m = 1.86\sigma_d$, $2.78\sigma_d$, $5.16\sigma_d$, $7.23\sigma_d$. The dashed (black) line corresponds to the bulk flow curve obtained in wide geometry using Lees–Edwards’ procedure.

**Fig. 10** Velocity profiles as a function of the position for a confinement $w = 40a$, and a wall stress $\sigma_w = 3.5\sigma_d$ and for two types of walls: smooth walls (red ●) and rough walls (black ●). We observe slipping for smooth walls. To compare the two velocity profiles we plot $V - V_{\text{max}}$ where $V_{\text{max}}$ is the maximum velocity at the center of the channel.
the stress is at the origin of another source of heterogeneity of the flow. Indeed an inhomogeneous stress leads to a spatially varying bulk fluidity, \( f_{\text{bulk}}(\sigma(z)) \), which is another source of non-locality in the system.

As previously the integration of the simplified eqn (17) is straightforward and leads to:

\[
\frac{\Gamma}{(d_1/d_2)^2 \sigma_{\text{a}}^2 \xi^2} = A \cosh(y/\xi) - e^{(y/\xi)} F(y) \tag{9}
\]

where

\[
F(y) = \begin{cases} 
0 & \text{if } |y| \leq y_d \\
\frac{1}{\xi^2} \int_{y_d}^{|y|} du e^{-2u/\xi} \left( \int_{y_d}^u dv v^{1/2} \Gamma_b \right) & \text{if } |y| > y_d
\end{cases}
\]

\( y_d \) is the position in the channel corresponding to a stress equal to \( \sigma_d \); \( y_d = L_0 \sigma_d / \Delta P \). The coefficient \( A \) is obtained from the boundary conditions. As previously we assume a given fluidity function at the wall, in the form \( \Gamma(w/2) = \Gamma_{\text{wall}}(\sigma_{\text{wall}}) \). One then deduces:

\[
A = \frac{\Gamma_{\text{wall}}(\sigma_{\text{wall}}) - e^{y_d/\xi} F(w/2)}{\cosh(w/2)} \tag{10}
\]

Following the same procedure as for the confined Couette cell, one then obtains the shear-rate profile, \( \dot{\gamma}(y) = f(y) \times \Delta P / L \), with \( f(y) = \frac{6}{G_0} \Gamma(y) \) the fluidity. The velocity is deduced by spatial integration of \( \dot{\gamma}(y) \).

Typical profiles of the rearrangement rate \( \Gamma(y) \) are plotted for different confinements for the same wall stress \( \sigma_w \) [Fig. 12(a)]. Several remarks can be made. In the absence of non-local effects, the profiles should overlap for all confinements. This is not the case here and this demonstrates that the viscosity does not simply depend on the local stress. Furthermore the deviation from the local rheology is no more restricted to the regions close to the wall only, as for the Couette case. This is particularly visible at the center of the cell, where the model predicts that the fluidity is non-zero at the center even if the stress is below the yield stress. Rearrangements in the zones of high stresses induce rearrangements in the zone of low stresses. This is a second key point of the KEP fluidity model. The velocity profile, plotted in Fig. 12(b), is less flat than expected for classical yield stress fluids. Furthermore, as for the fluidity profile, the velocity profiles do not overlap when the confinement is changed while keeping the wall stress \( \sigma_w \) constant. This is again a signature of the non-unique relation between the viscosity and the stress resulting from non-locality.

Using these results, we can now compare the simulation results. We fit the velocity profiles by integrating the KEP fluidity model, using \( f_{\text{bulk}} \) obtained from the simulations in wide geometry and \( \xi = 5R_s \) as for the confined Couette flow. The wall fluidity is extracted from the simulations. It varies between 1.5 and 2.5 \( \xi_{\text{bulk}} \), which is consistent with the value measured in Couette geometry. The results are displayed in Fig. 7(b). A very good agreement is found between the simulation results and the fluidity predictions, showing that the non-local rheology description captures the behavior of the investigated confined jammed material in the various geometries considered.

Further, in Fig. 13, we explore the variation in velocity profiles (and the corresponding fluidity profiles) with changing confinement – we show data for the periodic Poiseuille flow at a fixed \( \sigma_w \) for different values of channel-width \( w \). One can clearly observe that the velocity profiles (shown in the left panel of

\[ Fig. 12 \] (a) Rearrangement rate profiles obtained in Poiseuille geometry prediction from the KEP fluidity equations with \( \sigma_w = 5\sigma_d \). From top to bottom \( w = 10R_s, 20R_s, 30R_s, 50R_s \). (b) Reduced velocity profiles \( \nu/vw \) as a function of the reduced position predicted by the KEP fluidity equation using the same parameters as in (a). From top to bottom \( w = 8R_s, 10R_s, 20R_s, 40R_s \).

\[ Fig. 13 \] (a) Velocity profiles obtained in a periodic Poiseuille flow for different channel widths (\( w = 10R_s, 20R_s, 30R_s, 50R_s \)) for an imposed shear of \( \sigma_w = 1.89\sigma_d \) and (b) corresponding profiles for fluidity. Here, \( \sigma_w \) corresponds to the stress at the boundary between the two flow directions."
Fig. 13) do not overlap and in fact the shape changes with decreasing $w$. For example, in the case of $w = 10R_w$, the central plug-region shrinks and the profiles become more rounded. This is reflected in the fluidity profiles (shown in the right panel of Fig. 13); with decreasing $w$, we start to observe larger values of fluidity at the centre of the channel, exactly as discussed above. Also, note that in the case of a periodic Poiseuille flow, a fixed $\sigma_w$ (which is the stress at the boundary between the two flow directions) does not imply the same value of fluidity at the boundary – this was earlier demonstrated in the work by Chaudhuri et al.\textsuperscript{17} while studying confinement-dependence of dynamic arrest in this kind of flow. Non-locality coupled to the periodicity of the flow determines the shape of the fluidity profiles, linking the value at the boundary to that at the centre.

7 Microscopic insights: fluidity, velocity fluctuations and rearrangements

In the previous sections, we have mainly focused on the mean velocity profiles in different geometries. From these profiles, we have deduced that simulations exhibit non-local behavior in agreement with the predictions of the KEP fluidity model. We can try to go further by exploring the dynamics of the flowing material at the particle scales. Indeed, as emphasized previously, a key prediction of the KEP model is the intimate link between the fluidity, defined rheologically as the inverse viscosity $f = \gamma/\sigma$, and the rate of plastic rearrangements $\Gamma$. The KEP model predicts a linear relationship between the two, in the form $f = 6\Gamma/G_0$.

Here, we explore this question by analyzing the relationship between fluidity, rearrangement rates, as well as velocity fluctuations.

To measure the rearrangement rate we use the following procedure. From the centers of the particles we construct the Delaunay triangulation\textsuperscript{44} using a pre-programmed Matlab algorithm [Fig. 14]. This triangulation gives us the direct neighbors of each particle. One rearrangement is defined each time a link between two particles disappears. Some rearrangements may include more than two particles and in order to avoid overcounting, each time a link between two particles is broken, we do not count the rearrangements which may happen to their neighbors. This method gives the number of rearrangements per unit of time and per unit of surface. By multiplying by the surface of an elementary box $\sim R_w^2$, we deduce the local rearrangement rate $\Gamma$. We also calculate the local standard deviation of the instantaneous velocity $\langle \Delta V \rangle = \sqrt{\langle (V_i - \langle V \rangle)^2 \rangle}$.

This quantity is representative of the velocity fluctuations, which one expects to be linked to the rearrangement rate, as explored experimentally by Jop et al.\textsuperscript{18}

Fig. 15 shows the profiles of the rearrangement rate and of the standard deviation of the instantaneous velocity for different forcing. From this figure, one may first observe that both profiles exhibit the same features. First, they show a minimum at the center, while strongly increasing close to the wall, in qualitative agreement with the predictions of the KEP fluidity model [Fig. 12]. A small decrease is observed close to the wall for the velocity fluctuations, which corresponds to the first layer of particles. We also observe that the rearrangement rate and the velocity fluctuations are non-zero at the center, confirming that we observe non-vanishing dynamics in the system in the region where the stress is below the yield stress.

This motivated us to explore more specifically the link between the local fluidity and the micro-dynamics, characterized by the rate of plastic events and velocity fluctuation. As before the fluidity profile is extracted from the simulations from the ratio $f = \gamma/\sigma$: for the Poiseuille flow, the local stress $\sigma$ is known from mechanical equilibrium (linear in the distance from the channel center), while the local shear rates are obtained from spatial derivatives of the measured velocity profiles. In Fig. 16 and 17, we plot the (local) velocity fluctuations and the (local) rate of plastic events as a function of the (local) fluidity. A key finding exhibited in this plot is that all data for different external forcings or different confinements do collapse on a master curve.

We find an affine relation between the velocity fluctuations and the fluidity in the form: $\langle \Delta v \rangle \propto f$, see Fig. 17. This is consistent with the linear relation observed experimentally by Jop et al.\textsuperscript{18} On the other hand, data for the fluidity $f$ versus rearrangement rates $\Gamma$ collapse on a power–law relationship, with an exponent smaller than 1: $\Gamma \propto f^\alpha$ with $\alpha \approx 0.4$, Fig. 16. We recall that the KEP fluidity model predicts a linear relationship between the fluidity and the rate of events. While the numerical simulations and KEP fluidity model differ on the scaling function, the existence of a master relationship between

![Fig. 14](https://example.com/fig14.png) **Fig. 14** Typical configuration of a particle and the Delaunay triangulation constructed from this configuration.

![Fig. 15](https://example.com/fig15.png) **Fig. 15** Profile of rearrangement rate (left) and velocity fluctuation (right) as a function of the position for a Poiseuille flow and a confinement $w = 60R_w$ (the data are extracted from the simulation corresponding to the velocity profiles in Fig. 7). The wall stresses are from bottom to top $\sigma_w = 2\sigma_w, 2.5\sigma_w, 3.25\sigma_w$. As profiles are symmetric to the center, only half of the profiles are represented for clarity.
For the confinement \( w = 40R_i \), the data correspond to the following wall stress: (blue \( \bullet \)) \( \sigma_w = 2.75\sigma_d \), (red \( \blacksquare \)) \( \sigma_w = 3\sigma_d \), (magenta \( \bullet \)) \( \sigma_w = 3.25\sigma_d \), (green \( \Delta \)) \( \sigma_w = 3.5\sigma_d \), (blue \( \times \)) \( \sigma_w = 3.75\sigma_d \), (green \( \star \)) \( \sigma_w = 4\sigma_d \), (blue \( \triangle \)) \( \sigma_w = 4.25\sigma_d \) For the confinement \( w = 60R_i \), the data correspond to the following wall stress: (blue \( \bigcirc \)) \( \sigma_w = 2\sigma_d \), (red \( \square \)) \( \sigma_w = 2.25\sigma_d \), (magenta \( \bigtriangleup \)) \( \sigma_w = 2.5\sigma_d \), (green \( \blacklozenge \)) \( \sigma_w = 2.75\sigma_d \), (black \( \times \)) \( \sigma_w = 3\sigma_d \), (green \( \bigstar \)) \( \sigma_w = 3.25\sigma_d \).

the data correspond to the following wall stress: (blue \( \bullet \)) \( \sigma_w = 2.75\sigma_d \), (red \( \blacksquare \)) \( \sigma_w = 3\sigma_d \), (magenta \( \bullet \)) \( \sigma_w = 3.25\sigma_d \), (green \( \Delta \)) \( \sigma_w = 3.5\sigma_d \), (blue \( \times \)) \( \sigma_w = 3.75\sigma_d \), (green \( \star \)) \( \sigma_w = 4\sigma_d \), (blue \( \triangle \)) \( \sigma_w = 4.25\sigma_d \) For the confinement \( w = 60R_i \), the data correspond to the following wall stress: (blue \( \bigcirc \)) \( \sigma_w = 2\sigma_d \), (red \( \square \)) \( \sigma_w = 2.25\sigma_d \), (magenta \( \bigtriangleup \)) \( \sigma_w = 2.5\sigma_d \), (green \( \blacklozenge \)) \( \sigma_w = 2.75\sigma_d \), (black \( \times \)) \( \sigma_w = 3\sigma_d \), (green \( \bigstar \)) \( \sigma_w = 3.25\sigma_d \).

8 Conclusions and outlooks

In this work we have performed a detailed comparison between molecular dynamic simulations for a model jammed system under flow and predictions for non-local rheology derived recently on the basis of a mesoscopic kinetic elasto-plastic (KEP) approach. The bulk flow curve (i.e. in an “infinite” geometry) of the material is described by a Herschel–Bulkley law with an exponent of 1/2. For confined systems, simulations exhibit non-local rheology, in the sense that the relationship between the stress and the shear-rate is non-local and the apparent viscosity at a given point (or its inverse, the fluidity) also depends on the fluidization of the material in its neighborhood. The behavior of the simulated system is fully captured by the KEP fluidity model, which introduces as key quantities the fluidity of the material and a flow cooperativity length. Velocity profiles measured in simulations of both Couette and Poiseuille confined geometries exhibit strong departures from the bulk prediction and the fluidity model reproduces quantitatively the finite-size effects. Furthermore an intimate relationship between the local fluidity, defined as the inverse viscosity, and the local rate of plastic events is evidenced in the simulations, which takes the form of a master curve relating these two quantities obtained for all flow conditions. The simulations point to a power-law master relationship between fluidity and the rate of plastic events, with an exponent smaller than unity, while the KEP model predicts an exponent equal to one. At this stage, we do not have a detailed understanding of this observation, and full numerical simulations using alternative microscopic models, as well as in 3D, are required to explore this point more exhaustively. However the demonstration of such a link between the macroscopic rheology and the microdynamics does confirm a key prediction of the KEP fluidity model.

Altogether the simulations confirm that the KEP fluidity model is a proper and solid framework to describe the flow of yielding materials. Our findings are in agreement with previous experimental measurements, as well as alternative simulations of such materials using Lattice–Boltzmann techniques or lattice models. We quote as a side remark that a simplified version of the KEP fluidity model, eqn (17), is sufficient to account quantitatively for the simulated velocity profiles. However, the model in its full version, recalled in eqn (14), is also able to capture more subtle phenomena, as the confinement dependence of the dynamical arrest.

These results open many perspectives. One challenging question is the description of the flow behavior at the boundaries. Beyond the description of slippage effects, the fluidity approach requires boundary conditions for the wall fluidity, a point which was not realized before and which has key and subtle consequences on the influence of the wall roughness on the flow of these materials in confined geometries. Further work in this direction is needed to obtain a comprehensive picture.

Appendix

Fluidity model

In this appendix we recall the main ingredients of fluidity models, to which we will compare the results of our numerical simulations.

The essence of this class of models is to describe the local state of the system by a single scalar variable, the fluidity \( f \), usually defined as the local relaxation rate of the stress.
anticipate that $f$ is also proportional to the number of plastic events by unit time. The evolution of fluidity results from the competition between the relaxation of dissipative systems towards a quiescent, disordered state and flow rejuvenation.$^{22,23}$ Relaxation rates in the system decrease during the spontaneous evolution of the pasty system, the system becoming slower and slower with time. By contrast, flow induces rearrangements which act as a seed for new events and thus increase the fluidity $f$ of the system, as a feedback mechanism. In fluidity models, this competition is accounted for by an equation for the fluidity, which supplements a constitutive equation for the stress, e.g. a Maxwell equation. Neglecting convective terms, the equation for the fluidity is written in a general form:$^{23}$

$$\sigma = \dot{\gamma}/f$$

$$-A(f) + R(f, \sigma, \dot{\gamma}) = 0$$

(11)

where $\sigma$ is the local shear stress and $\dot{\gamma}$ is the local shear rate of the system. The term $A(f)$ stands for the relaxation processes towards a quiescent state, while $R(f, \sigma, \dot{\gamma})$ accounts for the flow-induced rejuvenation. Specific expressions for $A$ and $R$ are then required to obtain detailed predictions for the flow behavior.$^{23}$ Such models were successfully tested against experimental investigations of granular flows.$^{45}$

Furthermore, an extension of the model to account for non-locality of the rheology was proposed by adding a diffusive-like term:$^{23}$

$$\sigma = \dot{\gamma}/f$$

$$-A(f) + R(f, \sigma, \dot{\gamma}) + m\Delta f = 0$$

(12)

where $m$ is a physical constant and the Laplacian accounts for the cooperativity of the flow. The idea is that if the system is fluidized in a localized region, it will induce stress fluctuations of its neighborhood and thus a locally higher rate of plastic rearrangements. Note that in this general form the equation of fluidity is close to the granular temperature equation obtained within granular hydrodynamics of inelastic beads:$^{46}$ the spatial diffusion terms account for ‘heat diffusion’ (according to Fourier law) and the aging term to the energy dissipation due to inelastic collisions which drives the granular material toward a state of zero temperature. A proper generalization of such fluidity models to dense granular flow has been recently proposed by Kamrin and Koval.$^{37}$

Recently, these phenomenological models have been rationalized on a more microscopic basis. A kinetic model for the elastoplastic dynamics of a jammed material has been proposed (hereafter coined as the ‘KEP model’), which takes the form of a nonlocal – Boltzmann-like – kinetic equation for the stress distribution function, in line with the pioneering work of Hébraud and Lequeux.$^{41}$ In the stationary state, the model predicts rheological equations which take the generic form described above.$^{24}$ First, the local (average) stress and shear rate are connected via fluidity $f$, as an inverse viscosity, in the form:

$$\sigma = f^{-1}\dot{\gamma}$$

(13)

In this model, the fluidity is shown to be proportional to the local rate of plastic events, $\Gamma = \beta G_0^{-1}\gamma$, with $G_0$ the elastic modulus and $\beta$ a numerical coefficient ($\beta = 6$ in the original KEP model).

Then the rate of plastic events, $\Gamma$ (or equivalently the fluidity $f$), is found to obey a non-local diffusive-like equation

$$D - \alpha \Gamma = m\Delta \Gamma = a_1\sigma_d(\sigma_d - \sigma)\Gamma + a_2\sqrt{\alpha_1\sigma_c}^2\Gamma^{1/2}$$

(14)

where $\sigma_c$ and $\sigma_d$ correspond to a static and dynamic yield stress (with $\sigma_d < \sigma_c \sim G_0$); $\alpha_1$ is the stress relaxation time and $a_1$, $a_2$ are numerical constants. As a side remark, we quote that $\sigma_d$, $\alpha$, $a_1$, $a_2$, $m$ are deduced from the kinetic equations in the KEP model.$^{24}$ We get $\alpha = \frac{\sigma_c^2}{2}; \sigma_d = \sigma_d(12a_0)^{-1/2}(1 - a_0)\alpha_1; a_1 = 48a_0^2; a_2 = 4a_0\alpha_1(1 - a_0); m = \frac{R_k\sigma_c^2m_0}{2}$. In the previous expressions $a_0$ and $m_0$ are constants given by the elastic propagator.

**Low cooperativity approximation**

In order to compare with numerical simulations and experiments, it proves useful to write a simplified version of the previous equation for the fluidity. Indeed in the absence of heterogeneity ($m = 0$), eqn (14) yields a so-called ‘bulk’ solution for $\Gamma \equiv \Gamma_b$:

$$\sigma \leq \sigma_d; \Gamma_b = 0$$

$$\sigma > \sigma_d; \Gamma_b = \left(\frac{A_1}{A_2}\right)^2(\sigma - \sigma_d)^2$$

(15)

In a homogenous ‘bulk’ situation, this fluidity expression predicts a Herschel–Bulkley law for $\dot{\gamma} \rightarrow 0$. Indeed, from the expression of $\Gamma_b(\sigma)$ in eqn (15) and $\sigma = G_0\beta \Gamma^{-1}\dot{\gamma}$, and assuming $\sigma \sim \sigma_d$, one obtains:

$$\sigma_{\text{bulk}} = \sigma_d[1 + (t_k\dot{\gamma})^{1/2}]$$

(16)

with $t_k = \frac{G_0}{\beta \sigma_d}(\frac{A_2}{A_1})^{1/2}$; typically $t_k \sim t_k$, the stress relaxation time. Note that the same results are obtained independently of the low heterogeneity assumption, i.e. using eqn (14) or (17).

Let us now come back to the non-local situation, with a non-vanishing $m$. For low heterogeneity, then $\Gamma \sim \Gamma_b$ in eqn (14) and we obtain a simplified set of equations

$$\sigma = G_0\beta \Gamma^{-1}\dot{\gamma}$$

$$\Delta \Gamma = \frac{1}{\beta \Gamma}(\Gamma - \Gamma_b) = 0$$

(17)

with $\beta = 6$ for the original KEP model. We recall that the fluidity $f$, as introduced in the generic fluidity models, is directly proportional to $\Gamma$ in the KEP model, with $f = 6\Gamma/G_0$. 

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Soft Matter, 2013, 9, 7489–7500 | 7499
In those equations we have introduced a cooperativity length $\xi$.

$$\xi = \sqrt{\frac{(2)^m}{A_1|\sigma_0 - \sigma|}}$$  \hspace{1cm} (18)

with the factor 2 for $\sigma > \sigma_0$, and 1 otherwise.

While the cooperativity length $\xi$ is expected to depend on the shear-stress, eqn (18), in practice it will prove sufficient to omit this dependency when fitting the velocity profiles. Accordingly, in order to simplify predictions of the model and propose analytical results, we will assume in the following that $\xi = \xi_0$. This simplified set of equations was shown to predict quantitatively experimental results for the flow of concentrated emulsions in confined geometry with a single fitting parameter $\xi_0$. Note however that the full complexity of fluidity equations in eqn (14) was required to account for the confinement dependence of the dynamical arrest, as discussed previously by Chaudhuri et al.\textsuperscript{17}

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**References**