Diffusing-wave spectroscopy for arbitrary geometries: numerical analysis by a boundary-element method

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We present a boundary-element-method numerical procedure that can be used to solve for the diffusion equation of the field autocorrelation function in any arbitrary geometry with various boundary and source properties. We use this numerical method to study finite-sized effects in a circular slab and the influence of the angle in a cone-plate geometry. The latter is also compared with exact analytical solutions obtained for an equivalent bidimensional geometry. In most cases the deviation from well-known predictions of the correlation function remains small. © 2001 Optical Society of America

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1. Introduction

Multiple light-scattering techniques such as diffusing-wave spectroscopy provide powerful tools for the study of the dynamic properties of turbid media. Remarkably, the simple measurement of the diffuse-photon intensity and of the corresponding intensity autocorrelation function permit detection of the motion of heterogeneities (for example, colloid particles, emulsion droplets, foam membranes, and sand grains) with high temporal and spatial resolution in such a way that it is possible to identify and quantify whether the nature of their motion is simply ballistic, diffusive, or much more complex. Spatial and temporal variations are linked together from a one-to-one comparison of the experimental temporal correlation function and the theoretical correlation function, which can be expressed as a function of the mean-squared displacement of the scattering sites. Unfortunately, it is rather difficult to obtain an expression of the theoretical correlation function in geometries other than that of a flat cell, and this is a major limitation on the potential range of applications of the technique.

The problem stems from a combination of two facts. First, the properties of the diffusely emitted light can depend crucially on the geometrical shape of the container. Second, theoretical approaches used in modeling light diffusion do not generally provide an analytical form of the correlation function except in simple cases such as that of the previously mentioned flat cell. To apply multiple-light-scattering techniques in arbitrary geometries, one must a priori develop a method that will permit the numerical prediction of the correlation function for any given geometry. Such a method would be useful, for example, in rheology experiments, in which the use of a cone-plate geometry is very common and dynamic information obtained at a mesoscopic level can be a rich complement to the macroscopic rheological measurements. Another interesting application worth mentioning lies in the medical study of biological tissues or liquids such as blood cells streaming in vessels. The method would also be useful for testing the effect on the measurements of a lack of parallelism in a flat-cell geometry, the effect of curvature in a Couette geometry, or other deviations caused by the finite dimensions of the sample.

In this paper, we present a numerical technique based on a boundary-element method (BEM) that permits the determination of the correlation function for a wide range of geometries, boundary conditions, and source properties. We briefly describe a theoretical approach that models the diffusion problem by a diffusion equation for the field autocorrelation function, and we explain how to solve this transport equation numerically, by using the main ideas of the BEM. Analytical solutions are presented for a bidimensional wedge geometry, as are BEM numerical predictions obtained for three-dimensional geome-
tries such as a circular slab and a cone plate. It will be useful to compare analytical predictions with the results obtained by use of the BEM technique.

2. Theoretical Model

In the traditional approach to describing the transport of photons in a turbid medium it is assumed that diffusing photons move with a characteristic mean free path $l_0^s$, are created after penetrating the medium to a characteristic depth $z_c l_0^s$, and vanish at a typical distance $z_c l_0^s$ away from the diffusing medium’s boundaries. In the simplest case, the corresponding continuum equation that describes photon transport is a diffusion equation for the diffuse-photon concentration, but it can also be written as a telegrapher’s equation when ballistic photon transport is also important. Solving the transport equation yields the probability density $P(s)$ for detecting a photon that has migrated a path length $s$. Here we write the path-length distribution as $P(s) = P_{na}(s) \exp(-s/l_0^s)$, where $l_0$ is the absorption length in the diffusing medium and $P_{na}(s)$ refers to the (non-normalized) path-length distribution if there were no absorption. Then the normalized electric field autocorrelation is given by the fundamental equation of diffusing-wave spectroscopy:

$$g_1(\tau) = \int_0^\infty P_{na}(s) \exp\left[ -s/l_0^s - s/(3l_0^s) k_0^2 \langle \Delta r^2(\tau) \rangle \right] ds,$$  \hspace{1cm} (1)

where $k_0$ is the wave vector of light and $\langle \Delta r^2(\tau) \rangle$ is the mean-squared displacement of the scattering particles. From Eq. (1) it can be seen that absorption and motion of scattering sites have a similar effect on the correlation function. Also, a simple expansion of Eq. (1) at small $\langle \Delta r^2(\tau) \rangle$ shows that the initial decay rate, or first cumulant, of $g_1$ is set by the average path length, whereas the curvature, or second cumulant, is set by the width of $P(s)$.

For later comparison, we give the solution of Eq. (1) for light diffusely transmitted through a slab of thickness $L$ and infinite lateral dimensions. For this geometry with plane-wave illumination and either plane or point detection, the well-known result is

$$g_1(x) = \frac{\sinh(z_c \sqrt{x}) + z_c \sqrt{x} \cosh(z_c \sqrt{x})}{T_{z_c}(1 + z_c^2 x) \sinh(L \sqrt{x}) + 2z_c \sqrt{x} \cosh(L \sqrt{x})},$$  \hspace{1cm} (2)

where $L = l_0^s/n$, $x = k_0^2 \langle \Delta r^2(\tau) \rangle$, and $T_{z_c} = (z_p + z_c)/(L + 2z_c)$ is the probability of diffuse transmission. Equation (2) with the assumption that $z_p = 1$ is often used for analysis of experimental data; doing so neglects effects of scattering anisotropy, the distribution of penetration depths, multiple reflections of the incident (unscattered) beam, and ballistic transport, all of which are important for $L < 10l_0^s$.

In this paper, we do not use the classic approach because the transport equation of diffuse photons takes the form of partial differential equations that involve both time and space derivatives, which are rather difficult to solve numerically and whose full time and space solution is needed for determining the autocorrelation function. Instead, we choose an alternative description in which the field autocorrelation function is derived directly from an equation of the correlation transport that is approximated as a diffusion equation. For a point source localized at position $r_s$ with a strength $Q$, the diffusion equation for correlation is

$$[\nabla^2 + K^2(\tau)]G_1(r, \tau) = -3 \frac{Q}{l_0^s} \delta^3(\mathbf{r} - \mathbf{r}_s),$$  \hspace{1cm} (3)

with

$$K^2(\tau) = \frac{1}{l_0^s} \left[ \frac{3l_0^s}{l_0^s} + k_0^2 \langle \Delta r^2(\tau) \rangle \right].$$  \hspace{1cm} (4)

Equation (3) describes the correlation function in a stationary situation; hence it does not involve partial time derivatives and is a priori much simpler to solve numerically than the diffusion equation for the diffuse-photon concentration. Formally, it has the form of a Helmholtz equation, or, equivalently, a stationary diffusion equation with an absorption term. Physically, correlation diffuses along with the photons, and the motion of scattering particles introduces a loss of correlation that corresponds to an absorption effect that was already apparent in Eq. (1). The correlation function obeys physical boundary conditions that are formally identical to the one obtained for the diffusion equation for the diffuse-photon density in a stationary regime. For a medium with weak absorption the boundary condition can thus be written as

$$(z_c l_0^s \mathbf{n} \cdot \nabla + 1)G_1(\mathbf{r}) = 0,$$  \hspace{1cm} (5)

where $\mathbf{n}$ is the outer normal to the boundary. This boundary condition states that the linear extrapolation of $G_1$ from the boundary vanishes at a distance $z_c l_0^s$ outside the sample. In practice, this boundary condition is often approximated by instead requiring that $G_1$ itself vanish at the same distance.


The solution of partial differential equations by a standard numerical technique such as the finite-difference or the finite-element method requires a discretization of the whole three-dimensional volume. One of the main advantages of the BEM is that only the boundaries of this volume need be discretized, requiring a surface-modeling approach rather than volume modeling. In addition, the treatment of source terms is carried naturally from the exact mathematical definition of idealized point sources. We present a brief review of the technique, which has been described in much more detail by Brebbia et al. and by Brebbia and Dominguez. The BEM is based on an integral formulation of the differential equation. In the case of diffusion equa-
tion (3) the starting point is the Green’s theorem applied to the Helmholtz operator $H = \nabla^2 + K^2$
\[
\int_{\Omega} (gHW - wHG) d\Omega = \int_{\Gamma} \left( g \frac{\partial w}{\partial n} - w \frac{\partial g}{\partial n} \right) d\Gamma, \tag{6}
\]
where $\Omega$ represents a bounded volume, $\Gamma$ is the surface that bounds this volume, and, by definition, $\partial / \partial n = \mathbf{n} \cdot \mathbf{\nabla}$. The function $g$ is chosen to be the correlation function $G_1$ that satisfies a diffusion equation of the form $H_g = -\delta^2(r - r_i)$ in the volume that corresponds to the geometry of interest. For simplicity in writing the diffusion equation, we have assumed that there is a unique point source located at position $r_i$, and we have set the amplitude of the source term equal to unity, but it is straightforward to generalize the method to an arbitrary distribution of point sources. The function $w$ is chosen such that $HW = -\delta^2(r - r_i)$ in a domain of infinite volume. In other words, the function $w$ represents the fundamental solution of the problem in an infinite space for a point source located at position $r_i$. In what follows, it will become apparent that $w$ corresponds to the position at which the correlation function is determined. Indeed, as a result of our choice for functions $g$ and $w$, Eq. (6) can now be written as
\[
g(r_i) = w_{\mathcal{E}}(r_i) + \int_{\Gamma} \left( w_r \frac{\partial g}{\partial n} - g \frac{\partial w_r}{\partial n} \right) d\Gamma. \tag{7}
\]
It is crucial to realize that this relation now implicitly contains the condition that $g$ is a solution of the diffusion equation with proper boundary conditions and source properties. Furthermore, the analytical solution of the diffusion equation verified by $w_{\mathcal{E}}$ is known. In three dimensions it is simply expressed as
\[
w_{\mathcal{E}}(r) = \frac{\exp(-Kr - r_i)}{4\pi(r - r_i)}. \tag{8}
\]
Hence Eq. (7) shows that, to compute the value of $g$ at any point $r_i$ in volume $\Omega$, it is necessary only to know $g$ and its derivative $\partial g / \partial n$ at bounding surface $\Gamma$. In what follows, we detail the core procedure of the BEM that makes use of Eq. (7) to solve numerically for approximate values of $g$ and $\partial g / \partial n$ on boundary $\Gamma$.

The first step of the method is to approximate the surface integral that appears in Eq. (7). For this purpose the bounding surface $\Gamma$ is split into $N$ surface elements $\Gamma_i$ on which the functions $g$ and $\partial g / \partial n$ are approximated by use of a set of $p$ interpolation points located on element $\Gamma_i$, and of interpolation functions $\phi_{ji}(r)$ such that
\[
g(r_i) = \sum_{j=1}^{p} g_{ji} \phi_{ji}(r_i),
\]
\[
\frac{\partial g}{\partial n}(r) = \sum_{j=1}^{p} \left( \frac{\partial g}{\partial n} \right)_j \phi_{ji}(r).
\]
Here $g_{ji}$ and $(\partial g / \partial n)_j$ are constants that represent the value of the functions $g(r)$ and $\partial g / \partial n(r)$ at an interpolation point $r_i$ of surface element $\Gamma_i$. The shape of the surface element is arbitrary but is usually chosen as a rectangle or a triangle, and the number of points per surface element depends on the approximation level chosen in the interpolation procedure. The simplest case is that of $\Psi = 1$ and $p = 1$, meaning that $g$ and $\partial g / \partial n$ are considered constant over each element $\Gamma_i$, and can thus be defined by the values $g_i$ and $(\partial g / \partial n)_i$ that they take at an arbitrary point $r_i$ of each element $\Gamma_i$. More complex interpolation schemes are possible but necessitate a careful choice of the position of the interpolation points to overcome errors caused by the discontinuity of the derivative $(\partial g / \partial n)$ at the junction between two elements. In practice, elements of constant value are the easiest to implement and give surprisingly good results. From now on, we consider that the interpolation functions are constants and that the position of interpolation point $r_i$ on each element is the centroid of the element vertices.

The second step of the method consists of applying Eq. (7) to each point $r_i$ that pertains to boundary element $\Gamma_i$. Doing this requires careful integration because it is clear that each function $w_{\mathcal{E}}$ has a singularity at point $r_i$. If $\Gamma_i$ is a semispherical extension of the boundary centered about the point $r_i$, with a radius $\varepsilon$ the following limits are verified by the boundary integrals:
\[
\lim_{\varepsilon \to 0} \int_{\Gamma_i} w_r \frac{\partial g}{\partial n} d\Gamma_i = 0,
\]
\[
\lim_{\varepsilon \to 0} \int_{\Gamma_i} g \frac{\partial w_r}{\partial n} d\Gamma_i = -\frac{1}{2} g(r_i). \tag{9}
\]
Taking into account the singularity of integral terms about point $r_i$ and the approximation of the bounding surface by $N$ surface elements, we find an approximation of Eq. (7):
\[
\frac{1}{2} g_i = w_{\mathcal{E}}(r_i) + \sum_{k=1}^{N} \left[ \frac{\partial g}{\partial n} \right]_k \int_{\Gamma_k} w_r d\Gamma_k - g_k \int_{\Gamma_k} \frac{\partial w_r}{\partial n} d\Gamma_k. \tag{10}
\]
Inasmuch as $w_{\mathcal{E}}$ is a known function, the remaining integral terms in this approximate equation are easily computed, either analytically or numerically. Writing Eq. (10) for each one of the $N$ points $r_i$, we obtain a system of $N$ linear equations with $2N$ unknowns, $g_k$ and $(\partial g / \partial n)_k$. For each point $r_i$ boundary condition (5) becomes
\[
\int_{\Gamma_i} \frac{\partial g}{\partial n} + g_i = 0. \tag{11}
\]
From Eq. (9), half of the unknowns can be eliminated, and the linear system of \( N \) equations obtained from Eq. (8) can now be solved.

4. Analytical Results

To gain some insight into the ways in which the correlation function depends on the angle in a cone-plate geometry, we first derived an exact solution for a bidimensional wedge, which is the bidimensional equivalent of the cone-plate geometry, and for a bidimensional slab, which is the limit geometry obtained when the wedge angle becomes small. In these two geometries analytical expressions can be obtained relatively easily as long as the source is chosen to be a point. Generally, experiments are obtained relatively easily as long as the source is a point. Generally, experiments are made with either a plane-source–point-detection configuration or a point-source–plane-detection configuration. Here, for simplicity, the analytical estimates are made for a point-source–point-detection configuration. This choice should not change the qualitative behavior that we are trying to achieve too much.

A. Green’s Function for a Wedge

Consider the two-dimensional (2-D) wedge in polar coordinates \( 0 < \phi < \theta \) and \( r > 0 \), where \( \theta \) is the angle of the wedge. Let us denote the coordinates of a point source as \((r_s, \phi_s)\). In polar coordinates the Helmholtz equation becomes

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial g}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 g}{\partial \phi^2} + K^2 g = -\frac{\delta(r - r_s)\delta(\phi - \phi_s)}{r}.
\]

Instead of applying boundary condition (5), which is valid at the actual boundary of the diffusing medium, we use the approximate boundary condition \( g = 0 \) along the wedge sides, meaning that the actual boundaries are at a distance \( z_p l^b \) inside the wedge. With the following choice of boundary condition, \( g(r, 0) = 0 \) and \( g(r, \theta) = 0 \), the equation has an analytical solution in the form of an infinite series that reads as\(^{10}\)

\[
g(r, \phi) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{4}{ab} \frac{\sin(m \pi x/a) \sin(m \pi x_s/a) \sin(n \pi y/b) \sin(n \pi y_s/b)}{K^2 + (m^2 \pi^2/a^2) + (n^2 \pi^2/b^2)}
\]

\[
\times \frac{\sin(m \pi x/a) \sin(m \pi x_s/a) \sin(n \pi y/b) \sin(n \pi y_s/b)}{K^2 + (m^2 \pi^2/a^2) + (n^2 \pi^2/b^2)}.
\]

where 0 \( \leq x \leq a \) and 0 \( \leq y \leq b \). The analytical solution for a 2-D finite slab can be used as a double check that the 2-D wedge solution correctly converges toward the 2-D slab solution when the wedge angle tends toward zero.

C. Numerical Estimates

We used Eq. (11) to obtain a numerical estimate of the wedge solution. It is important to remember here that the boundaries along which we imposed \( g = 0 \) are not the physical boundaries but are assumed to be at a distance \( z_p l^b \) from the physical boundaries. This distance is, in practice, almost constant and comparable with \( l^b \), so we assume throughout the rest of the paper that \( z_p = 1 \). The distance between the point source \( S \) and the boundary, \( \phi = 0 \), is chosen equal to \( (z_p + z_p)l^b \). Similarly, \( z_p \) is, in practice, close to unity, so from now on we set \( z_p = 1 \). The position \( M \) at which we are looking at the correlation function is such that \( MS \) is perpendicular to the boundary, \( \phi = 0 \), and is on the hypothetical physical boundary at a distance \( z_p l^b \) from the boundary \( \phi = 0 \). In addition, when we change the wedge angle, we maintain as fixed the local thickness \( L \) of the wedge and choose the radius of the point source such that its distance from the measured correlation point is always \( L - z_p l^b \). This choice ensures that we compare correlation functions that correspond to the same equivalent local thickness, whatever wedge angle \( \theta \) is.

All the results shown in this paper are for \( L/l^b = 10 \). We used Eq. (12) to obtain a numerical estimate of the 2-D slab solution with a ratio of \( a/b = 10 \), where \( b = L + 2z_p l^b \). The point source is placed halfway across the slab width at \( x_s = a/2 \) and at a height of \( y_s = (z_p + z_p)l^b \). The correlation function is computed at position \( x = a/2, y = L + z_p l^b \). As for the wedge, we choose \( L/l^b = 10 \). Results are shown in Fig. 1(a).

First, we note that, with a decreasing angle, the 2-D wedge solution tends toward the 2-D slab solution, although the 2-D slab has finite dimensions. It is also apparent from this figure that the variations of the correlation functions with the wedge angle stay rather small. However, the correlation function is slightly more curved for higher angles of the wedge, and the limit of a 2-D slab is approached from higher values of the correlation function.

A qualitative explanation for this effect can be given: Let us assume that the boundary closest to the source point is fixed and that the distance from the source point to the measuring point is fixed too. Then the boundary at which the measurement is made will make an angle \( \theta \) with respect to the fixed...
boundary. For a slab geometry \( \theta = 0 \) the accessible scattering area is distributed symmetrically with respect to a direction perpendicular to the boundaries and passing through the point source. For the wedge \( \theta \neq 0 \) the accessible scattering area is distributed asymmetrically with respect to a direction perpendicular to the fixed boundary and passing through the source point. Nevertheless, a typical slice of scattering surface about the detection point has the same area in the slab and the wedge geometry. For this reason, we do not expect the wedge angle to cause a significant change in the average path length. However, the increase in the scattering area on the thicker side of the wedge will have a tendency to shift the path-length distribution \( P(s) \) toward longer path lengths \( s \), whereas on the narrower side the decrease in the scattering area will shift \( P(s) \) toward smaller \( s \). Overall, the distribution \( P(s) \) is expected to broaden slightly compared with that in the slab situation about the same average path length. From Eq. (1), we saw that the initial curvature of the correlation is set by the width of the path-length distribution \( P(s) \). Thus the broadening of the distribution \( P(s) \) implies an increase in the correlation-function curvature, and the net effect is a global increase of the correlation function as the wedge angle becomes wider.

![Image](image_url)

**Fig. 1.** (a) Correlation functions of the 2-D wedge and the 2-D slab for transmitted light for point-source illumination and point detection as obtained from analytical solutions of the 2-D diffusion equation. \( \theta \) is the wedge angle, \( L \) is the sample thickness at the detection point, and \( K^2 = (L/\Delta s)(D/L + k_0^2\Delta r(t)) \), according to Eq. (4). (b) Corresponding effective transport mean free path plotted as a function of wedge angle. Note that, for wedge angles less than 10°, the 2-D slab result holds to within approximately 0.1%.

Let us try to quantify further the difference between the correlation functions obtained for the 2-D slab and the wedge. Remember that, from Eq. (4), in the absence of an absorption length, \( K^2 = k_0^2(\Delta r^2(t))/L^2 \). As \( k_0 \) and \( \langle \Delta r^2(t) \rangle \) are given parameters, \( L^2 \) seems to be the only adjustable parameter that could, in a way, encompass the effect of geometry changes. Then we can try to scale the curves and look for an effective mean free path \( l_{\text{eff}}^* \) as in Ref. 11. This is only a first-order approximation and does not take into account deviations in the functional form of the solution. Figure 1(b) shows the effective mean free path \( l_{\text{eff}}^* \) scaled by the mean free path \( l_0^* \) for the 2-D slab. The relative increase in \( l_{\text{eff}}^* \) remains smaller than 2% for angles as wide as 45°.

5. Results of the Boundary-Element Method

A. Finite Size Effect in a Circular Slab

Before we investigate the cone-plate geometry, which is the ultimate goal of our paper, it is instructive to test the BEM predictions in a situation for which there is already some experimental insight. Finite size effects have already been observed by Kaplan et al.\textsuperscript{11} in the case of slab geometry. They used a circular slab of diameter \( D \) and thickness \( L \) that was lit with a circular spot perpendicularly to one of the circular sides. Instead of changing the aspect ratio \( D/L \) of the slab, they changed the diameter of the circular spot \( D_\text{s} \) and found that a ratio \( D_\text{s}/L \) of at least 5 was necessary for recovery of the infinite-slab plane-wave predictions. The geometry that we consider is essentially the same. However, in our case the circular slab is always lit entirely across the circular section as the diameter of the slab is varied.

We implemented the BEM by using a plane source that is modeled as a mesh of ideal point sources with a characteristic spacing of the order of \( l_0^* \). In practice, surface modeling of the geometry and of the source has been done with the help of commercial software.\textsuperscript{12} The slab thickness is chosen to be \( L = 10l_0^* \), and the slab diameter is varied. In Fig. 2(a) the correlation functions predicted by the BEM are displayed for several values of the aspect ratio \( D/L \), and we also show for comparison the infinite-slab solution. For \( D/L \geq 6 \) the finite slab and the infinite slab are virtually indistinguishable. Once again, we can try to scale the curves by choosing the infinite slab as a reference. Figure 2(b) shows the effective mean free path \( l_{\text{eff}}^* \) scaled by the infinite-slab mean free path \( l_{\text{inf}}^* \) plotted as a function of the aspect ratio \( D/L \). The finite size effect is important at small aspect ratios, but above an aspect ratio of roughly \( D/L = 6 \) the effect clearly becomes negligible. The results obtained with the BEM are thus qualitatively the same as those observed experimentally, and we can now approach the cone-plate geometry with more confidence.

Incidentally, we have shown that cutting off the long photon paths by reduction of the lateral dimensions of the circular slab leads to a global increase in the correlation function. This is a simpler effect...
than that observed for the 2-D wedge. For the circular slab the average path length is obviously reduced when the aspect ratio becomes small, which in turn has a direct effect on the correlation function by reduction of the initial decay rate. When the aspect ratio becomes larger than, typically, \( D/L = 5 \), the cutoff occurs at so long a path length that there is virtually no further change in the path-length distribution. Then the correlation function obtained for an infinite slab becomes an extremely good estimate.

B. Cone Plate

We obtained predictions of the BEM for a cone plate by considering the case of a plane source and the physical boundary-condition equation (5). As for the circular slab, the characteristic spacing of the idealized point sources that form the plane source is set to be comparable with photon mean free path \( l^* \). An example of the mesh is shown in Fig. 3. The correlation function is measured at a point located on the cone surface, at mid-distance along the radius of the circular plate. The plane-source position is set at a distance \( z = l^* \) from the plate (recall that \( z_p = 1 \)) and such that the distance from the measuring point to the plane source is equal to \( 9l^* \), making the local thickness of the cone-plate geometry equal to \( L = 10l^* \).

In Fig. 4(a) the correlation function is shown for several values of the angle between the cone and the plate. The solid curve shows the asymptotic behavior for an infinite slab. The behavior at a high angle closely resembles the one observed in two dimensions; i.e., the correlation function exhibits a greater curvature, which decreases as the angle decreases. An unexpected effect occurs at intermediate angles at which the correlation function becomes less curved before it reaches the asymptotic limit. This result can be more easily visualized from Fig. 4(b), which shows the effective photon mean free path scaled by

![Fig. 2](http://example.com/fig2.png)

![Fig. 3](http://example.com/fig3.png)

![Fig. 4](http://example.com/fig4.png)
the infinite-slab mean free path \( l_{\text{inf}}^* \) as a function of the cone-plate angle. Near an angle of 20° there is clearly an effect that is opposite that of the 2-D wedge effect and that tends to decrease the correlation-function curvature beyond the limit obtained when the angle becomes null.

As for the 2-D wedge results and the circular slab, it is possible to understand qualitatively the surprising behavior that we observed for the cone-plate geometry: If we consider a typical volume about the detection point, where the local thickness is \( L \), the portion of this volume that is accessible to the photons is larger for cone-plate geometry than for a slab of the same thickness \( L \). This is not an obvious result and deserves a numerical estimate. We describe the cone-plate geometry in cylindrical coordinates \((r, \alpha, z)\) and write \( R \) for the radius of the plate and \( 2L \) for the thickness of the plate at radius \( R \). With this choice the local thickness at the detection point is \( L \) when the plate’s radius is \( R/2 \). Let us take a portion of the cone-plate geometry centered about the detection point such that \( R/2 - \Delta R < r < R/2 + \Delta R \) and \(-\alpha_0/2 < \alpha < \alpha_0/2\), where \( \alpha_0 R/2 = 2\Delta R \). The scattering volume thus delimited is

\[
V_s = 4L(\Delta R)^2 \left[ 1 + \frac{1}{3} \tan^2 \theta(\Delta R/L)^2 \right].
\]

The volume of the equivalent slab that has the same horizontal section and thickness \( L \), however, is simply

\[
V_s = 4L(\Delta R)^2.
\]

Comparing the two expressions, we see that the delimited cone-plate volume is equivalent to the volume of a slab with an effective thickness

\[
L_{\text{eff}} = L \left[ 1 + \frac{1}{3} \tan^2 \theta(\Delta R/L)^2 \right]. \tag{13}
\]

From Eq. (13) it becomes clear that the average path length in the cone-plate geometry increases as the cone angle increases, which explains why the correlation functions obtained for small angles are below the infinite-slab solution. As for the 2-D wedge, we also expect a broadening of the path distribution, but here this is a secondary effect, specifically, note that deviations from \( l_{\text{inf}}^* \) shown in Fig. 4(b) are much larger than those shown in Fig. 1(b). We previously tried to quantify our results by introducing an effective mean free path \( l_{\text{eff}}^* \), while keeping \( L \) constant, and here we introduce an effective thickness \( L_{\text{eff}} \), while keeping \( l^* \) fixed. To reconcile the two points of view, we set \( L/l_{\text{eff}}^* = L_{\text{eff}}/l^* \), which gives us an analytical estimate of the effective mean free path. The best fit to the numerical data yields \( \Delta R/L = 1.02 \pm 0.03 \). Most simply, taking \( \Delta R/L = 1 \), we have the following result for the effective transport mean free path in a cone-plate geometry:

\[
l_{\text{eff}}^* = \frac{l_{\text{inf}}^*}{1 + \frac{1}{3} \tan^2 \theta}, \tag{14}
\]

which is shown as a solid curve in Fig. 4(b). Despite some slight discrepancy, it works remarkably well, and it also confirms that the typical scale that defines the scattering volume is simply \( L \).

Although the analysis presented above may explain the numerical results for small angles, it does not capture the behavior of the correlation function at large angles. Remember that the detection radius is half of the geometry radius. For small angles of the cone plate, the radius is much larger than \( L \), so photons can never reach the outer boundary or the cone apex. However, for large angles, the sample radius becomes comparable to \( L \), so photons start to leak out the side. As for the finite size effect observed for the circular slab with \( D/L < 4 \), the average path length decreases and the correlation functions decay less rapidly. At intermediate angles there is a competition between an increase in the scattering volume and a finite size effect.

6. Conclusion

We hope that this study has provided more insight into the effect of geometrical properties for the application of diffusing-wave techniques. Most of the time, geometry has a rather negligible effect on the technique, and, when it has a larger effect, it is often possible to consider that geometry simply introduces a new effective photon mean free path. In some instances, such as for the cone-plate geometry, the effect is nontrivial, but it remains quite small. Nevertheless, the BEM numerical approach that we have developed remains a powerful tool for analyzing a great variety of experimental configurations. Only the case of a medium with homogeneous diffusing properties has been addressed, but a full development of the dual-reciprocity BEM\(^1\) should permit nonhomogeneous materials to be considered.

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References and Notes